Topology and its Applications 28 (1988) 113-125
North-Holland

ON COLLECTIONWISE HAUSDORFFNESS IN COUNTABLY PARACOMPACT, LOCALLY COMPACT SPACES

Peg DANIELS*
Division of Mathematics, FAT, Auburn University, AL 36849, U.S.A.

Received 1 April 1985
Revised 10 December 1985

We obtain various results related to the question whether countably paracompact, locally compact, metacompact (or screenable) spaces are CWH, and hence (as we show) paracompact.

AMS (MOS) Subj. Class.: Primary 54D15, 54D18; secondary 54A35

metacompact, screenable, collectionwise Hausdorff, countably paracompact, collectionwise normal, paracompact, sparse set

1. Introduction

Many authors have investigated whether or not normality can be replaced by countable paracompactness[1] in collectionwise Hausdorff (CWH) or collectionwise normal (CWN) results; generally the answer has been 'yes', although the proofs of positive results are often more complicated (see e.g. [22], [5], [6], [19], [20] and [2]).

An interesting open question due independently to Arhangel'skii and Tall is the following: is every normal, locally compact, metacompact space paracompact? Watson has shown the answer is "yes" assuming $V = L$ [22], and we have shown it to be "yes" if "metacompact" is strengthened to "boundedly metacompact" [7]. (See [1], [12], [13] for other assumptions under which the answer is "yes".) There are no known counterexamples under any set-theoretic assumptions. It is well-known that to decide the question it suffices to decide whether every normal, locally compact metacompact space is CWN w.r.t. compact sets, or in fact whether every such space is CWH. There is also no known counterexample to Arhangel'skii's and Tall's

* Work done while a postdoctoral fellow at University of Toronto.
[1] See the next section for definitions.

question if "normal" is weakened to "countably paracompact" (here "weakened" is appropriate since a normal (countably) metacompact space is countably paracompact). It is not surprising that it suffices to decide whether every countably paracompact, locally compact, metacompact space is CWN w.r.t. compact sets (or CWH), although the proof that this is the case is more difficult than in the 'normal' analogue and depends upon the fact that such spaces are the increasing union of countably many boundedly metacompact spaces; we include the proof of this fact in this paper. To prove this reduction we actually only need to show that every locally compact, boundedly metacompact space that is CWN w.r.t. compact sets is paracompact.

This leads us to investigate under what conditions locally compact, boundedly metacompact spaces are CWN w.r.t. compact sets. It is often useful in trying to prove CWN (CWH) to proceed by induction on λ, showing that if a space is $<\lambda$-CWN w.r.t. compact sets ($<\lambda$-CWH), then it is λ-CWN w.r.t. compact sets (λ-CWH). In [7] we showed every normal, locally compact, boundedly metacompact space is CWH (hence paracompact) in this manner. In trying to get the analogous result when "normal" is replaced by "countably paracompact", we realized neither property is necessary to show that $<\lambda$-CWH implies λ-CWH for singular λ; we thus show the following: if X is locally compact, boundedly metacompact, and $<\lambda$-CWN w.r.t. compact sets for λ singular, then X is λ-CWN w.r.t. compact sets. The proof is an application of Fleissner's notion of "sparse" sets.

The problem thus lies with regular λ; we do not have any absolute results for this case (unless, of course, the space is normal), but we show that the induction goes through for regular λ in countably paracompact such spaces assuming $V=L$. Balogh later showed that assuming $V=L$, every countably paracompact, locally compact space is CWH, and hence every metacompact such space is paracompact [2]. Although this supercedes our L results, the techniques we develop and present here, due to their combinatorial nature, may be useful in determining whether these results are absolute. Also, they have led to further results: in [8] we show that countably paracompact, locally compact, boundedly metacompact spaces are paracompact, assuming $\mathrm{PFA}^+(1)$.

Finally, it turns out that there is a perhaps unexpected relationship between countably paracompact, locally compact, boundedly metacompact spaces and countably paracompact, locally compact, screenable spaces: namely, we show that if all the former are CWH, then all the latter are paracompact (and thus the latter are paracompact assuming $V=L$, $\mathrm{PFA}^+(1)$, etc.). In [3], Balogh shows in ZFC that all normal, locally compact, screenable spaces are paracompact.

Local compactness is essential in the results on screenable spaces: Rudin has shown that assuming $\diamond^{++}$, a combinatorial principle which holds in L, there is a normal, screenable, nonparacompact space [17], and has pointed out that Wage's machine for producing countably paracompact, nonnormal spaces from perfectly normal, noncollectionwise normal spaces [21] takes a non-CWH space to a screenable space; so applying Wage's machine to Bing's example H [4] gives a countably paracompact, screenable, nonnormal, non-CWH space [18]. (Note this is one setting in which countable paracompactness does not behave like normality.)

2. Definitions

We gather here the definitions of various concepts that have appeared in the introduction and will appear throughout this paper.

A space X is:

CWH ($<\lambda$-*CWH*, λ-*CWH*) provided that every closed discrete set (of size $<\lambda$, of size $\leqslant\lambda$) can be separated;

CWN (*CWN w.r.t. compact sets*) provided that every discrete collection of closed (compact) sets can be separated;

(*countably*) *paracompact* provided that every (countable) open cover has a locally finite open refinement;

metacompact provided that every open cover has a point-finite open refinement;

boundedly metacompact (*n-boundedly metacompact*) provided that every open cover has an open refinement of finite order (order n);

screenable provided that every open cover has a σ-disjoint open refinement.

The following definitions are also needed:

the character of a point $x\in X$ is the minimum cardinality of a base at x;

the *character of a set* $K\subset X$ is its character in the quotient space obtained by identifying K to a point.

3. Paracompactness in countably paracompact, locally compact, metacompact spaces

In this section we show that every countably paracompact, locally compact, (sub)metacompact space that is CWN w.r.t. compact sets is paracompact, and, therefore, if every countably paracompact, locally compact, (boundedly) metacompact space is CWH, then every such space is paracompact. Recall that if X is normal, locally compact, metacompact, and CWN w.r.t. compact sets, then X is paracompact. If we strengthen "metacompact" to "boundedly metacompact", we can drop "normal" and still have a theorem of ZFC; it turns out that to prove the first result mentioned above, it suffices to prove the result for 0-dimensional boundedly metacompact spaces. A couple of preliminary lemmas will prove useful.

Lemma 3.1. *If* $f:X\to Y$ *is perfect,* X *is* 0*-dimensional, and* Y *is* (n-) *boundedly metacompact then* X *is* (n-) *boundedly metacompact.*

Proof. Assume the hypothesis and suppose $\mathcal{U}$ is a cover of X by clopen sets. $\{Y\backslash f(X\backslash\bigcup\mathcal{U}')\colon \mathcal{U}'\in[\mathcal{U}]^{<\omega}\}$ is an open cover of Y, so let $\mathcal{W}$ be an open refinement of this collection of order n for some integer n. (We will simultaneously be taking care of the case that Y is n-boundedly metacompact.) For each $W \in \mathcal{W}$, let $\mathcal{U}_W \in [\mathcal{U}]^{<\omega}$ be such that $W \subset Y \setminus f(X \setminus \bigcup \mathcal{U}_W)$; since each such $\mathcal{U}_W$ is a finite collection of clopen sets, find a finite disjoint collection $\mathcal{V}_W$ of clopen sets which refines $\mathcal{U}_W$ and is such that $\bigcup \mathcal{U}_W = \bigcup \mathcal{V}_W$. $\mathcal{V} = \{f^{-1}(W) \cap V\colon W \in \mathcal{W}$ and $V \in \mathcal{V}_W\}$ is an open refinement of $\mathcal{U}$ and has order n: suppose $x \in X$ is in $n+1$ distinct elements of $\mathcal{V}$, say $f^{-1}(W_0) \cap V_0, f^{-1}(W_1) \cap V_1, \ldots, f^{-1}(W_n) \cap V_n$. Since $f(x) \in \bigcap_{m \leq n+1} W_m$, there must be two integers j, $k \leq n+1$ such that $W_j = W_k$; since V_j and V_k belong to the disjoint collection $\mathcal{V}_{W_j}$, $V_j = V_k$. This is a contradiction, so $\mathcal{V}$ must have order n, and so X is (n-) boundedly metacompact. □

(The hypothesis that X is 0-dimensional is necessary—consider the disjoint union of the spaces I^n for each $n \in \omega$; it is not boundedly metacompact, yet the map that collapses each I^n to a point is certainly perfect and gives a discrete space. This is not surprising in view of the work done by Fletcher, McCoy, and Slover [12] which demonstrates the intimate relationship between bounded metacompactness and finite dimension.)

Another useful fact is due to Gruenhage and Michael [15]:

The GM Lemma. *If $\mathcal{V}$ is an open cover of a regular space X by open sets with Lindelöf boundaries then there is a closed cover $\mathcal{A} = \{A_V\colon V \in \mathcal{V}\}$ of X such that $A_V \subset V$.*

Corollary 3.2. *If X is 0-dimensional, locally compact, and boundedly metacompact, then X is n-boundedly metacompact for some n.*

Proof. If X is 0-dimensional and $\mathcal{V}$ is an open cover of X by sets with compact closures, then there is a collection $\mathcal{A}$ as in the GM Lemma which consists of (compact) clopen sets. Thus if X is 0-dimensional, boundedly metacompact, and locally compact, we can get a clopen, compact cover of X of order n for some integer n, by starting out with a cover by open sets with compact closures, refining it by an open cover of order n for some n, by the bounded metacompactness of X, and then applying the above observation. In [7], however, we showed that if a space can be covered by compact, clopen sets of order n, it is the perfect pre-image of an n-boundedly metacompact space, and so by Lemma 3.1 it is n-boundedly metacompact. □

The last preliminary lemma is the following;

Lemma 3.3. *If X is 0-dimensional, locally compact, boundedly metacompact, and λ-CWN w.r.t. compact sets, then every open cover of X of size $\leq \lambda$ by sets with compact closures has a disjoint clopen refinement.*

Proof. Suppose X is as in the hypothesis. By Corollary 3.2, let $n \in \omega$ be such that X is n-boundedly metacompact. Suppose $\mathcal{U} = \{U_\alpha\colon \alpha < \lambda\}$ is an open cover of X

by sets with compact closures; without loss of generality, $\mathcal{U}$ consists of compact clopen sets and is of order n. First suppose $n=2$. For each α, let $H_\alpha = U_\alpha \setminus \bigcup_{\beta\neq\alpha} U_\beta$. $\{H_\alpha: \alpha<\lambda\}$ is a discrete collection of compact sets, so let $\{V_\alpha: \alpha<\lambda\}$ be a separation of this collection; since each H_α is compact, there is a clopen set W_α such that $H_\alpha \subset W_\alpha \subset V_\alpha$, so without loss of generality we may assume each V_α is clopen, and $V_\alpha \subset U_\alpha$. Since $\mathcal{U}$ has order 2,

$$\{V_\alpha: \alpha<\lambda\} \cup \{U_\alpha \cap U_\beta)\setminus(V_\alpha \cup V_\beta): \alpha\neq\beta \text{ in } \lambda\}$$

is a disjoint open refinement of $\mathcal{U}$. Now suppose $n>2$ and the lemma is true for $(n-1)$-boundedly metacompact spaces. Again define $\mathcal{H}_1=\{H_\alpha: \alpha<\lambda\}$ and $\mathcal{V}_1=\{V_\alpha: \alpha<\lambda\}$ as above. Now suppose $j<n-1$ and H_F and V_F have been defined for each $F\in[\lambda]^k$, $k\leqslant j$ (here for $k=1$ we are identifying $H_{\{\alpha\}}$ with H_α and $V_{\{\alpha\}}$ with V_α). For each $k\leqslant j$, let $\mathcal{V}_k=\{V_F: F\in[\lambda]^k\}$. For $F\in[\lambda]^{j+1}$, let $H_F=\{x\in X: \{\alpha<\lambda: x\in U_\alpha\}=F\}\setminus\bigcup_{k<j}(\bigcup \mathcal{V}_k)$. Then $\{H_F: F\in[\lambda]^{j+1}\}$ is a discrete collection of compact sets of size λ, so let $\mathcal{V}_{j+1}=\{V_F: F\in[\lambda]^{j+1}\}$ be a clopen separation of this collection with $V_F\subset\bigcap_{\alpha\in F} U_\alpha$.

Now $\bigcup_{j\leqslant n-1}(\bigcup \mathcal{V}_j)$ is clopen in X: if x is not in this union, then $\bigcap\{U_\alpha\setminus\bigcup\{V_F: F\in[\{\beta: x\in U_\beta\}]^{\leqslant n-1}\}: x\in U_\alpha\}$ is an open set containing x and missing this union, since $\mathcal{U}$ has order n and V's refine the corresponding U's. Let $Y=\bigcup_{j\leqslant n-1}(\bigcup \mathcal{V}_j)$; Y is 0-dimensional and covered by a collection of compact clopen sets of order $n-1$ (namely $\bigcup_{j\leqslant n-1}\mathcal{V}_j$), and hence is $(n-1)$-boundedly metacompact. By the induction hypothesis, $\bigcup_{j\leqslant n-1}\mathcal{V}_j$ has a disjoint clopen refinement, say $\mathcal{W}$. Then $\mathcal{W}\cup\{\bigcap_{\alpha\in F} U_\alpha\setminus Y: F\in[\lambda]^n\}$ is a disjoint clopen refinement of $\mathcal{U}$. □

Since every regular space is the perfect image of a 0-dimensional space, it can be shown by modifying standard techniques and using the GM lemma that if X is locally compact, boundedly metacompact, and λ-CWN w.r.t. compact sets, then X is λ-paracompact w.r.t. open covers by sets with compact closures, i.e. every open cover of X of size $\leqslant\lambda$ by sets with compact closures has a locally finite open refinement (see [10], section 5.1]).

Theorem 3.4. *If X is locally compact, boundedly metacompact, and CWN w.r.t. compact sets, then X is paracompact; furthermore, if X is also 0-dimensional, then it is ultraparacompact (every open cover has a disjoint open refinement).*

Proof. Assume X is as in the hypothesis; without loss of generality we may assume X is 0-dimensional. Every open cover of X has an open refinement by sets with compact closures, and since X is λ-CWN w.r.t. compact sets for each λ, by Lemma 3.3 such a refinement has a disjoint clopen refinement. Thus X is (ultra-) paracompact. □

We do not know whether bounded metacompactness can be weakened to metacompactness in the above theorem.

To show every countably paracompact, locally compact, submetacompact space that is CWN w.r.t. compact sets is paracompact, we show that every such (0-dimensional) space is the increasing union of closed boundedly metacompact spaces, which are then paracompact by the previous theorem.

Theorem 3.5. *All countably paracompact, locally compact, submetacompact spaces that are CWN w.r.t. compact sets are paracompact.*

Proof. Suppose X has all the properties mentioned in the hypothesis. Let f be a perfect map from a zero-dimensional space Y onto X. Then Y is countably paracompact, locally compact, submetacompact [6] and CWN w.r.t. compact sets. We proceed much as in the proof of the 'normal' analogue to this theorem. Let $\mathscr{U}$ be an open cover of Y by compact clopen sets, and let $\langle \mathscr{V}_n \rangle_{n\in\omega}$ be a sequence of open refinements of $\mathscr{U}$ such that for each $x \in X$ there is an $n \in \omega$ such that x is in only finitely many elements of $\mathscr{V}_n$; using the GM Lemma we assume without loss of generality that the elements of each $\mathscr{V}_n$ are compact clopen. For each $V \in \mathscr{V}_0$ let $H_v = \{x \in V\colon$ there is no $U \in \mathscr{V}_0 \setminus \{V\}$ such that $x \in U\}$. $\{H_v\colon V \in \mathscr{V}_0\}$ is a discrete collection of compact sets, and so we may let $\mathscr{W}_0 = \{W_V\colon V \in \mathscr{V}_0\}$ be a clopen compact separation of this collection. Now continue on, much as in the proof of Lemma 3.3, by considering all the points that are in exactly two elements of $\mathscr{V}_0$ but not in $\bigcup \mathscr{W}_0$, etc. Repeat this process for each $\mathscr{V}_n$, and in this way, generate a σ-disjoint cover of X by clopen compact sets, $\mathscr{R} = \bigcup_{n\in\omega} \mathscr{R}_n$. For each $n \in \omega$, let $R_n = \bigcup_{m\leq n} (\bigcup \mathscr{R}_n)$. Since the R_n's form an increasing sequence of open sets with union Y, by one characterization of countable paracompactness [16], we may let F_n be a closed set such that $F_n \subset R_n$ and $\bigcup_{n\in\omega} F_n^\circ = Y$. Note that each F_n is a zero-dimensional space that can be covered by a collection of compact clopen sets of order n, and so F_n is (n-) boundedly metacompact. By Theorem 3.4 each F_n is paracompact, and so by standard arguments, Y is paracompact. Since paracompactness is preserved by closed maps, X is paracompact. □

We now use a standard quotient argument to get:

Theorem 3.6. *If every countably paracompact, locally compact,* (*boundedly*) *metacompact space is CWH, then every such space is CWN w.r.t. compact sets* (*and hence, paracompact*).

Proof. Mimic the usual proof in which "normal" replaces "countably paracompact" by taking such a space X and a discrete collection of compact sets and collapsing each of these compact sets to a point. The resulting quotient space is also countably paracompact and metacompact by well-known results; an analysis of Theorem 6 in [7] shows that the natural quotient map resulting from collapsing a discrete collection of closed (not necessarily compact) sets preserves (n-) bounded metacompactness if the domain space is regular, so the hypothesis can be applied to the quotient space to get a separation of the closed discrete set; bringing the separation back to the domain space gives a separation of the discrete collection. □

Theorems 3.5 and 3.6 are well-known for the case where "normal" replaces "countably paracompact".

4. CWH-ness in countably paracompact, locally compact, boundedly metacompact spaces

We now investigate the hypothesis of Theorem 3.6 by using two different approaches. First we show:

Theorem 4.1. *If X is locally compact, boundedly metacompact, and $<\lambda$-CWN w.r.t. compact sets for singular λ, then X is λ-CWN w.r.t. compact sets.*

Next we show that to decide whether the hypothesis of Theorem 3.6 is true, it suffices to decide it for 0-dimensional 2-boundedly metacompact spaces. We then show that this gives a result on countably paracompact, locally compact, screenable spaces. Fleissner's notion of sparseness can be used in the proof of Theorem 4.1. As the proof is quite technical, we make it easier for the reader by first presenting some definitions and results that are useful for the proof. including a character reduction lemma.

The following definition and theorem are from [11]:

Definition. Suppose $Y\subset X$ is a closed discrete set of cardinality $\lambda>\operatorname{cf}\lambda$. Identify Y with λ and let $\langle\lambda_\alpha\colon \alpha<\operatorname{cf}\lambda\rangle$ be an increasing continuous sequence of cardinals cofinal in λ, with $\lambda_0\geqslant \operatorname{cf}\lambda,\ \omega_2$. Y is *sparse* provided for each continuous increasing sequence $\langle A_\alpha\colon \alpha<\operatorname{cf}\lambda\rangle$ such that $A_\alpha\subset\lambda$, $|A_\alpha|=\lambda_\alpha$, and $\bigcup_\alpha A_\alpha=\lambda$, there is a neighborhood assignment $\mathcal{U}$ to the points of Y such that for each α, $|\bigcup\{U(\beta)\colon \beta\in A_\alpha\}\cap\lambda|=\lambda_\alpha$.

Sparse Set Theorem. *If Y is sparse and $<\lambda$-separated, then Y is separated.*

It is easy to change the definition of "sparse" to the following to have the above theorem still hold:

Definition. Suppose $Y\subset X$ is a closed discrete set of cardinality $\lambda>\operatorname{cf}\lambda$. Identify Y with λ. Y is *sparse* provided that there is a continuous increasing sequence of infinite cardinals cofinal in λ, $\langle\lambda_\alpha\colon \alpha<\operatorname{cf}\lambda\rangle$, such that for each continuous increasing sequence $\langle A_\alpha\colon \alpha<\operatorname{cf}\lambda\rangle$ with $A_\alpha\subset\lambda$, $|A_\alpha|=\lambda_\alpha$, and $\bigcup_\alpha A_\alpha=\lambda$, there is a neighborhood assignment $\mathcal{U}$ to the points of Y such that for each α, $|\bigcup\{U(\beta)\colon \beta\in A_\alpha\}\cap\lambda|=\lambda_\alpha$.

The following 'character reducing' lemma implies that if X is 0-dimensional, locally compact, and metacompact, and H is an unseparated discrete collection of compact sets in X of size λ, then there is a closed $Y \subseteq X$ and an unseparated discrete collection of compact sets in Y of size λ, each element of which has character λ; if X is boundedly metacompact, then so is Y. In later proofs that use this lemma we will need to see how Y is constructed, and so we state the lemma in greater generality. We will use the following notations and definitions:

Suppose X is a space and $\mathcal{U} = \{U_\alpha: \alpha < \kappa\}$ is a collection of open sets. For each $x \in X$, let $F^x = \{\alpha < \kappa: x \in U_\alpha\}$. For $\alpha \neq \beta$ in κ and $x \in X$, call F^x a *min set for* α, β iff $\{\alpha, \beta\} \subset F^x$ and there is no $y \in X$ such that $\{\alpha, \beta\} \subset F^y \subsetneq F^x$. Let $\Lambda(\lambda) = \lambda \cup \bigcup\{F^y: y \in X$ and there are α, β in λ such that F^y is a min set for $\alpha, \beta\}$, for each cardinal λ. For each cardinal $\lambda \leqslant \kappa$, let $Z(\lambda) = \{y \in X: F^y \subset \Lambda(\lambda)\}$, and for each $\alpha < \lambda$ let $G_\alpha^\lambda = \{y \in Z(\lambda): F^y = \{\alpha\}\}$.

Lemma 4.2 (Character Reducing Lemma). *Suppose X is a space, $\mathcal{C} = \{C_\alpha: \alpha < \lambda\}$ is a discrete collection of compact sets in X, and $\mathcal{U} = \{U_\alpha: \alpha < \kappa\}$ is a point finite cover of X by clopen, compact sets such that for each $\alpha \in \lambda$, $C_\alpha \subset U_\alpha$ and if $\beta \in \kappa \backslash \{\alpha\}$, $C_\alpha \cap U_\beta = \emptyset$. Then for each $\lambda \leqslant \kappa$*

(a) *$Z(\lambda)$ is a closed subspace of X containing $\mathcal{C}$, and $\{G_\alpha^\lambda: \alpha < \lambda\}$ is a discrete collection of compact sets in $Z(\lambda)$ such that for each $\alpha < \lambda$, $\{(U_\alpha \backslash \bigcup_{\gamma \in F} U_\gamma) \cap Z(\lambda): F \in [\Lambda(\lambda) \backslash \{\alpha\}]^{<\omega}\}$ is a base for G_α^λ in $Z(\lambda)$, and so G_α^λ has character $\leqslant |\Lambda(\lambda)|$ in $Z(\lambda)$;*

(b) *$\Lambda(\lambda)$ has cardinality λ;*

(c) *$\{G_\alpha^\lambda: \alpha < \lambda\}$ cannot be separated in $Z(\lambda)$ if $\{C_\alpha: \alpha < \lambda\}$ cannot be separated in X.*

Proof. Suppose X, $\mathcal{C}$, and $\mathcal{U}$ are as in the lemma.

(a) $Z(\lambda)$ is closed since $z \notin Z(\lambda)$ iff $F^z \not\subset \Lambda(\lambda)$ iff $\bigcap_{\alpha \in F^z} U_\alpha \cap Z(\lambda) = \emptyset$; $\bigcup \mathcal{C} \subset Z(\lambda)$ since for each $\alpha < \lambda$ and $x \in C_\alpha$, $F^x = \{\alpha\}$ and $\alpha \in \Lambda(\lambda)$. It is easy to check that $\{G_\alpha^\lambda: \alpha < \lambda\}$ is a discrete collection of closed sets in $Z(\lambda)$, and since G_α^λ is a subset of U_α, G_α^λ is compact.

We now show that $\{(U_\alpha \backslash \bigcup_{\gamma \in F} U_\gamma) \cap Z(\lambda): F \in [\Lambda(\lambda) \backslash \{\alpha\}]^{<\omega}\}$ is a base for G_α^λ in $Z(\lambda)$. By way of contradiction, suppose U is open in $Z(\lambda)$, contains G_α^λ, and for each $F \in [\Lambda(\lambda) \backslash \{\alpha\}]^{<\omega}$, $z_F \in (U_\alpha \backslash (\bigcup_{\gamma \in F} U_\gamma \cup U)) \cap Z(\lambda)$. Let $A = \{z_F: F \in [\Lambda(\lambda) \backslash \{\alpha\}]^{<\omega}\}$; A is a compact subset of $(U_\alpha \backslash U) \cap Z(\lambda)$, and for each $x \in A$ we may let $\alpha_x \in \Lambda(\lambda) \backslash \{\alpha\}$ such that $x \in U_{\alpha_x}$, since otherwise $F^x = \{\alpha\}$ in which case $x \in G_\alpha \subset U$. Since A is compact, there must be a finite subset of $\{\alpha_x: x \in A\}$, call it F, such that $\{U_\beta: \beta \in F\}$ covers A, but then z_F is an element of A not covered by this collection, a contradiction.

Clearly it follows that each G_α^λ has character $\leqslant |\Lambda(\lambda)|$ in $Z(\lambda)$.

(b) It suffices to show that for each $\alpha \neq \beta$ in λ, $\not{f}_{\alpha,\beta} = \{F^y: y \in X$ and F^y is a min set for $\alpha, \beta\}$ is finite, since for each $y \in X$, F^y is finite. Since $\mathcal{U}$ is point-finite, $\{\bigcap_{\gamma \in F^x} U_\gamma: x \in X$ and F^x is a min set for $\alpha, \beta\}$ is an open cover of $U_\alpha \cap U_\beta$, a compact set. Let $\{x_j: j \leq k\} \subset X$ be such that each F^{x_j} is a min set for α, β, and $\{\bigcap_{\gamma \in F^{x_j}} U_\gamma: j \leq k\}$ covers $U_\alpha \cap U_\beta$ $(k \in \omega)$. Suppose $\gamma \in \bigcup \mathscr{f}_{\alpha,\beta} \setminus \bigcup_{j \leq k} F^{x_j}$, and let $y \in X$ be such that F^y is a min set for α, β and $\gamma \in F^y$. But since there must be a $j \leq k$ such that $y \in \bigcap_{\delta \in F^{x_j}} U_\delta$, $F^{x_j} \subsetneq F^y$, contradicting the fact that F^y is a min set for α, β.

(c) Suppose $\mathscr{G} = \{G^\lambda_\alpha: \alpha < \lambda\}$ can be separated in $Z(\lambda)$. For each $\alpha < \lambda$ let $F_\alpha \in [A(\lambda) \setminus \{\alpha\}]^{<\omega}$ be such that $\{(U_\alpha \setminus \bigcup_{\gamma \in F_\alpha} U_\alpha) \cap Z(\lambda): \alpha < \lambda\}$ is a separation of $\mathscr{G}$ in $Z(\lambda)$. We claim that this collection is a separation of $\mathscr{G}$, and therefore of $\mathscr{C}$, in X. On the contrary, suppose $p \in (U_\alpha \setminus \bigcup_{\gamma \in F_\alpha} U_\gamma) \cap (U_\beta \setminus \bigcup_{\gamma \in F_\beta} U_\gamma)$ for some $\alpha \neq \beta$ in λ. Since $\{\alpha, \beta\} \subset F^p$, we may let $y \in X$ such that F^y is a min set for α, β and $F^y \subset F^p$. Now $y \in Z(\lambda)$, and since $F^p \cap (F_\alpha \cup F_\beta) = \emptyset$, $F^y \cap (F_\alpha \cup F_\beta) = \emptyset$, which means y is also in this intersection, a contradiction. □

Note in the proof of (c) the essential use of the min sets to reflect in $Z(\lambda)$ the intersection properties in X of the cover.

If X, $\mathscr{C}$, and $\mathscr{U}$ are as in Lemma 4.2, let $Y(\lambda)$ denote the space obtained from $Z(\lambda)$ by collapsing each G^λ_α to a point, denoted by g^λ_α. Then $\{g^\lambda_\alpha: \alpha < \lambda\}$ is a closed discrete set in $Y(\lambda)$ that cannot be separated in X, and if q is the natural quotient map, each g^λ_α has $\{q((U_\alpha \setminus \bigcup_{\gamma \in F} U_\gamma) \cap Z(\lambda)): F \in [A(\lambda) \setminus \{\alpha\}]^{<\omega}$ as a local basis; in particular, each g^λ_α has character $\leq \lambda$ in $Y(\lambda)$.

In the proof of Theorem 4.1, we use Lemma 3.1 to assume without loss of generality that our spaces are 0-dimensional, thus n-boundedly metacompact by Corollary 3.2, and then proceed by induction on n. Lemma 3.3 is useful in the inductive step.

Proof of Theorem 4.1. Suppose N is locally compact, boundedly metacompact, and $<\lambda$-CWN w.r.t. compact sets for singular λ. Let $f: X \to N$ be perfect for some 0-dimensional X. Then X is locally compact; X is boundedly metacompact by Lemma 3.1, and in fact n-boundedly metacompact for some n by Corollary 3.2; in [9] we showed that CWN w.r.t. compact sets is preserved in the inverse image direction by perfect maps, and the same proof shows this is true for $<\lambda$-CWN w.r.t. compact sets—thus X is $<\lambda$-CWN w.r.t.

Suppose $\mathscr{C} = \{C_\alpha: \alpha < \lambda\}$ is a discrete collection of compact sets in X. Using the GM Lemma, there is a collection $\mathscr{U} = \{U_\alpha: \alpha < \kappa\}$ of clopen compact sets of order n covering X such that for each $\alpha < \lambda$, $C_\alpha \subset U_\alpha$ and if $\beta \in \kappa \setminus \{\alpha\}$, then $C_\alpha \cap U_\beta = \emptyset$. Using the notation preceding and following the Character Reducing Lemma, denote $A(\lambda)$ by A, $Z(\lambda)$ by Z, each G^λ_α by G_α, $Y(\lambda)$ by Y, and each g^λ_α by g_α. Let $G = \{g_\alpha: \alpha < \lambda\}$. Recall that for each α, $\{U_\alpha \setminus \bigcup_{\gamma \in F} U_\gamma: F \in [A \setminus \{\alpha\}]^{<\omega}\}$ is a base at G_α (and that the quotient images of these sets give a base at g_α). Let $\langle \lambda_\alpha: \alpha < \mathrm{cf}\,\lambda \rangle$ be a sequence of cardinals increasing up to λ, $\lambda_0 \geq \omega$. First suppose $n = 2$. We show that G is sparse in Y. Suppose $\langle A_\alpha: \alpha < \mathrm{cf}\,\lambda \rangle$ is a continuous increasing sequence such that $|A_\alpha| = \lambda_\alpha$, $\bigcup_\alpha A_\alpha = \lambda$. It suffices to show that for each $\alpha < \mathrm{cf}\,\lambda$, $|\bigcup \{q(U_\beta): \beta \in A_\alpha\} \cap G| = \lambda_\alpha$. Suppose not, and let $B \subset \lambda$ be such that $A_\alpha \subset B$, $\lambda_\alpha < |B| < \lambda$, and $\{g_\beta: \beta \in B\} \subset \bigcup \{q(U_\beta): \beta \in A_\alpha\}$. Now $\{g_\beta: \beta \in B\}$ can be separated in Y (since $\{G_\beta: \beta \in B\}$ is a discrete collection of $<\lambda$ compact sets in X and thus can be separated). So for each $\beta \in B$, let $F_\beta \in [\Lambda]^{<\omega}$ be such that $\{q(U_\beta \backslash \bigcup_{\gamma \in F_\beta} U_\gamma): \beta \in B\}$ is a separation of $\{g_\beta: \beta \in B\}$. Let $\beta \in B \backslash (A_\alpha \cup \bigcup \{F_\beta: \beta \in A\})$. Since $\beta \in B$, let $\delta \in A_\alpha$ be such that $q(U_\beta \backslash \bigcup_{\gamma \in F_\beta} U_\gamma) \cap q(U_\delta) \neq \emptyset$. Since $q(U_\beta \backslash \bigcup_{\gamma \in F_\beta} U_\gamma) \cap q(U_\delta \backslash \bigcup_{\gamma \in F_\delta} U_\gamma) = \emptyset$ and $\mathscr{U}$ is of order 2, we must have that $\beta \in F_\delta$, a contradiction. So G is sparse and $<\lambda$-separated, and thus by Fleissner's Sparse Set Theorem, G can be separated in Y, and thus $\mathscr{H}$ can be separated in X.

Now suppose that $n > 2$ and all spaces that are 0-dimensional, $(n-1)$-boundedly metacompact, locally compact, and $<\lambda$-CWN w.r.t. compact sets are λ-CWN w.r.t. compact sets. Our goal is to show Z is 2-boundedly metacompact.

Let $Z' = \{z \in Z: |\{\alpha \in \Lambda: z \in U_\alpha\}| \leqslant n-1\}$. Z' is $(n-1)$-boundedly metacompact, hence by the inductive assumption it is λ-CWN. By Lemma 3.3, $\{U_\alpha \cap Z': \alpha \in \Lambda\}$ has a disjoint open refinement, so let $\{V_\alpha: \alpha \in \Lambda\}$ be a collection of open sets in Z such that $\{V_\alpha \cap Z': \alpha \in \Lambda\}$ is such a refinement, with each $V_\alpha \subset U_\alpha$. Now $\{V_\alpha: \alpha \in \Lambda\} \cup \{\bigcup_{\beta \in F} U_\beta: F \in [\Lambda]^n\}$ is an open cover of Z by sets with compact closures, so using the GM Lemma we may assume without loss of generality that each V_α is compact clopen. Let g be a 1-1 function from λ onto Λ, and for each $\alpha < \lambda$, let $U'_\alpha = U_{g(\alpha)}$ and $V'_\alpha = V_{g(\alpha)}$. For each $\alpha < \mathrm{cf}\,\lambda$, let $A_{\alpha,0} = \lambda_\alpha$; given $A_{\alpha,n}$, let $A_{\alpha,n+1} = A_{\alpha,n} \cup \{\beta$: there are γ, δ in $A_{\alpha,n}$ such that $V'_\beta \cap V'_\gamma \cap V'_\delta \neq \emptyset\}$; let $A_\alpha = \bigcup_n A_{\alpha,n} \backslash \bigcup_{\beta<\alpha} A_\beta$. Let us show that $|A_\alpha| = \lambda_\alpha$ for each α. Suppose γ, $\delta \in \lambda$. Since $V'_\gamma \cap V'_\delta$ is a compact subset of $Z \backslash Z'$, it can meet only finitely many elements of $\{\bigcup_{\beta \in F} U_\beta: F \in [\Lambda]^n\}$, as this is a disjoint open cover of $Z \backslash Z'$, say $\{\bigcap_{\beta \in F_j} U_\beta: j \leqslant k\}$. If β is such that there is an element, say x, in $V'_\beta \cap V'_\gamma \cap V'_\delta$, then $V'_\gamma \cap V'_\delta \cap \bigcap_{\alpha \in F^x} U_\alpha \neq \emptyset$ and so $F^x = F_j$ for some $j \leqslant k$; furthermore, $x \in V_{g(\beta)} \subset U_{g(\beta)}$, and so $\beta \in g^{-1}(\bigcup_{i \leqslant k} F_i)$. From this it follows that $|A_\alpha| = \lambda_\alpha$ for each α, and so $\{V'_\beta \cap Z': \beta \in A_\alpha\}$, a discrete collection of compact sets in Z (and thus in X) of size $<\lambda$, can be separated in Z; let $\{W_\beta: \beta \in A_\alpha\}$ be such a separation by clopen sets, $W_\beta \subset V'_\beta$. We claim that $\{W_\beta: \beta \in \lambda\}$ has order 2. Suppose not, and let β, γ, $\delta \in \lambda$ and $z \in Z$ be such that $z \in W_\beta \cap W_\gamma \cap W_\delta$; let $\alpha(\beta)$, $\alpha(\gamma)$, $\alpha(\delta) \in \mathrm{cf}\,\lambda$ be such that $\beta \in A_{\alpha(\beta)}$, $\gamma \in A_{\alpha(\gamma)}$, $\delta \in A_{\alpha(\delta)}$; without loss of generality, assume $\alpha(\beta) < \alpha(\gamma) < \alpha(\delta)$; let $n(\beta)$, $n(\gamma) \in \omega$ be such that $\beta \in A_{\alpha(\beta),n(\beta)}$, $\gamma \in A_{\alpha(\gamma),n(\gamma)}$; but then β, $\gamma \in A_{\alpha(\gamma),n(\gamma)+n(\beta)}$ (since $A_{\phi,m} \subset A_{\theta,m}$ for $\phi \leqslant \theta$), and so $\delta \in A_{\alpha(\gamma)}$, a contradiction. Note that for each $F \in [\lambda]^n$, $\bigcap_{\alpha \in F} U'_\alpha \backslash \bigcup_{\alpha \in F} W_\alpha$ misses each W_β. Thus $\{W_\beta: \beta \in \lambda\} \cup \{\bigcap_{\alpha \in F} U'_\alpha \backslash \bigcup_{\alpha \in F} W_\alpha: F \in [\lambda]^n\}$ is a clopen compact of Z of order 2, and so Z is 2-boundedly metacompact. We have already seen that this means that $\{G_\alpha: \alpha < \lambda\}$ can be separated in Z, and thus $\mathscr{C}$ can be separated in X.

Thus X is λ-CWN w.r.t. compact sets, and it is easy to check that this implies N is λ-CWN w.r.t. compact sets. □

In the proof of Theorem 4.1, it was sufficient to prove the result for 0-dimensional, 2-boundedly metacompact spaces. By using similar techniques the following can be proved:

Lemma 4.3. *If every* 0*-dimensional, countably paracompact, locally compact,* 2*-boundedly metacompact space is CWH, then every countably paracompact, locally compact, boundedly metacompact space is CWH (and hence paracompact).*

Sketch of proof. Assume the hypothesis and suppose X is 0-dimensional, countably paracompact, locally compact, and boundedly metacompact; X is n-boundedly metacompact for some n by Corollary 3.2; if $n=2$, X is CWH, so let us assume $n>2$. Let us also assume that we have shown that λ is a cardinal such that every 0-dimensional countably paracompact, locally compact, boundedly metacompact space is $<\lambda$-CWH (and thus $<\lambda$-CWN w.r.t. compact sets) and that every 0-dimensional, locally compact, $(n-1)$-boundedly metacompact space is λ-CWH (and thus λ-CWN w.r.t. compact sets). If λ is singular, then by Theorem 4.1 X is λ-CWH, but even if λ is regular, $\lambda<\omega$, we can proceed exactly as in the case $n>2$ in the proof of Theorem 4.1 to show that X is λ-CWH (let $\lambda_\alpha=\alpha$ for each $\alpha\geqslant\omega$); since X is regular, it is ω-CWH. Thus we have the conclusion for every such 0-dimensional space, and hence every such 0-dimensional space is CWN w.r.t. compact sets. It is now easy to check that if Y is the perfect image of such a 0-dimensional space, then Y is CWN w.r.t. compact sets; thus the conclusion of Lemma 4.3 holds. □

It turns out that this lemma provides us with a result for countably paracompact, locally compact, screenable spaces:

Theorem 4.4. *If every* 0*-dimensional countably paracompact, locally compact,* 2*-boundedly metacompact space is CWH, then every countable paracompact, locally compact, screenable space is paracompact.*

Proof. Assume the hypothesis and suppose that X is a countably paracompact, locally compact, screenable space; without loss of generality assume X is 0-dimensional. Let $\mathcal{U}$ be an open cover of X by compact, clopen sets, and let $\mathcal{V}=\bigcup_{n\in\omega}\mathcal{V}_n$ be a σ-disjoint open refinement; by the GM Lemma, we may assume that the elements of each $\mathcal{V}_n$ are compact clopen. As in Theorem 3.5, let $R_n=\bigcup_{m\leqslant n}(\bigcup\mathcal{V}_m)$, and let F_n be closed such that $F_n\subset R_n$ and $\bigcup_{n\in\omega}F_n^\circ=X$. Each F_n is $(n$-$)$ boundedly metacompact, hence paracompact by the last lemma, hence X is paracompact. □

5. On countably paracompact, locally compact, 2-boundedly metacompact spaces

It may be a theorem of ZFC that all countably paracompact, locally compact, 2-boundedly metacompact spaces are CWH (hence paracompact), but we have only been able to show that this is true assuming $V=L$, that such spaces are $<\mathfrak{c}$-CWH assuming MA+$\neg$CH, and that various strengthenings of MA_{ω_1} imply that such spaces are CWH (and thus that every countably paracompact, locally compact, screenable space is paracompact). To avoid burdening the reader with the additional technicalities of the results involving MA and its strengthenings, these will be presented in [8]; here we present the $V = L$ result. We need the following result of Watson's [22]:

Assuming $\diamondsuit$ for stationary systems, if X is countably paracompact and $<\lambda$-CWH for λ regular, then X is λ-CWH for sets whose points have character $\leqslant\lambda$.

Theorem 5.1 ($V = L$). *If X is countably paracompact, locally compact, and* (2-) *boundedly metacompact, then X is CWH* (*hence paracompact*).

Proof. Since every space satisfying the conditions of the hypothesis is ω-CWH, let λ be the least uncountable cardinal such that every such space is $<\lambda$-CWH (hence $<\lambda$-CWN w.r.t. compact sets) but there is such a 0-dimensional space, call it X, that is not λ-CWH. Then by the Character Reducing Lemma (see the remarks following it) there is such a space Y with an unseparated closed discrete set, say $G = \{g_\alpha: \alpha < \lambda\}$, each point of which has character $\leqslant\lambda$. If λ is regular, however, Watson's result implies that G can be separated in Y (since $V = L$ implies $\diamondsuit$ for stationary systems) but if λ is singular, Theorem 4.1 implies that G can be separated in Y. We have a contradiction, so every such 0-dimensional space must be CWH, hence paracompact, and thus 0-dimensionality can be removed to get the statement of the theorem. $\square$

Corollary ($V = L$). *Every countably paracompact, locally compact screenable space is paracompact.*

The combinatorial nature of our approach to the problems studied in this paper can be made even more obvious by noting that all zero-dimensional, locally compact, metacompact spaces are perfect pre-images of subspaces of Pixley–Roy spaces on cardinals given the co-finite topology (see [7]). In particular, such 2-boundedly metacompact spaces are perfect pre-images of subspaces of the following types of spaces X:

Let λ be a cardinal and $\mathcal{I} \subset [\lambda]^2$. Let $X = \lambda \cup \mathcal{I}$ be topologized as follows: the points of $\mathcal{I}$ are isolated, and for each $\alpha \in \lambda$ and $F \in [\lambda\backslash\{\alpha\}]^{<\omega}$, a basic open set about α is $\mathcal{U}(\alpha, F) = \{\alpha\} \cup \{\{\alpha, \beta\} \in \mathcal{I}: \beta \notin F\}$ (this corresponds to the open set $U_\alpha \backslash \bigcup_{\beta \in F} U_\beta$ about G_α in the Character Reducing Lemma, etc.).

References

[1] A. Arhangel'skii, The property of paracompactness in the class of perfectly normal, locally bicompact spaces, Soviet Math. Doklady 13 (1972) 517–520.
[2] Z. Balogh, On countably paracompact, locally compact, submetacompact spaces, preprint.
[3] Z. Balogh, Paracompactness in locally Lindelöf spaces, preprint.

[4] R.H. Bing, Metrization of topological spaces, Can. J. Math. 3 (1951) 175-186.

[5] D.K. Burke, PMEA implies countably paracompact Moore spaces are metrizable, Proc. AMS. 92 (1984) 455-460.

[6] D.K. Burke, Closed mappings, in: Surveys in General Topology (Academic Press, New York, 1980) 1-32.

[7] P. Daniels, Normal, locally compact, boundedly metacompact spaces are paracompact: An application of Pixley-Roy spaces, Canad. J. Math. 35 (1983) 807-823.

[8] P. Daniels, When countably paracompact, locally compact, screenable spaces are paracompact, preprint.

[9] P. Daniels, Perfect pre-images of collectionwise normal spaces, Proc. AMS, to appear.

[10] R. Engelking, General Topology (Polish Scientific Publishers, Warsaw, 1977).

[11] W. Fleissner, Discrete sets of singular cardinality, Proc. AMS 88 (1983) 743-745.

[12] P. Fletcher, R.A. McCoy and R. Slover, On boundedly metacompact and boundedly paracompact spaces, Proc. AMS 25 (1970) 335-342.

[13] G. Gruenhage, Paracompactness and subparacompactness in perfectly normal, locally compact spaces, Russian Math. Surveys 35 (1980) 49-55.

[14] G. Gruenhage, Applications of a set-theoretic lemma, Proc. AMS 92 (1984) 133-140.

[15] G. Gruenhage and E. Michael, A result on shrinkable open covers, Top. Proc. 8 (1983) 37-43.

[16] F. Ishikawa, On countably paracompact spaces, Proc. Japan Acad. 31 (1955) 686-687.

[17] M.E. Rudin, A normal, screenable, nonparacompact space, Topology Appl. 15 (1982) 313-322.

[18] M.E. Rudin, Dowker spaces, in: Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 761-780.

[19] F. Tall, Countably paracompact Moore spaces are metrizable in the Cohen model, Top. Proc. 9 (1984) 145-148.

[20] F. Tall, Weakly collectionwise Hausdorff spaces, Top. Proc. 1 (1976) 295-304.

[21] M. Wage, Countable paracompactness, normality, and Moore spaces, Proc. AMS 57 (1976) 183-188.

[22] W.S. Watson, Separation in countably paracompact spaces, Trans. AMS 290 (1985) 831-842.